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FOCUSING OF SHOCK WAVES IN A HIGHLY VISCOUS FLUID*

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The method of combined asymptotic expansions is used to solve the problem of the focusing of a shock wave (in a weakly compressible medium of high viscosity. Asymptotic forms of the solution are constructed in a number of spatial zones. The focusing zone is described by its asymptotic form obtained by combining it with the solution corresponding to viscous geometrical acoustics. The reflection of a shock wave formed as a result of velocity jump near one of the foci of the ellipsoid of revolution is discussed as an example. Analytical relationships descrbing the focusing zone around the second focus are obtained. It is shown that at the focus itself the wave profile has an antisymmetric form, and the compression wave is followed by a rarefaction wave of the same form.

The problem of the formation and propagation of a spherically symmetrical shock wave in a highly viscous fluid was solved in /1/. The wave reflected from an ellipsoid also becomes spherical, although it is no longer spherically symmetrical.

The structure of the zone in which a shock wave of moderate intensity is focused can be described, if at least one of the following factors is taken into account, namely the non-linearity and/or dissipation. If both factors are disregarded, then the wave amplitude will be governed by the law of non-viscous geometrical acoustics up to the focal point, and the focusing zone will degenerate to a point /2/. In the case of fairly strong shock waves the curved front will straighten because of the non-linearity, and therefore the wave intensity may not increase strongly and the focusing zone will shift from the geometrical focus and will increase /3, 4/. In devices where exact focusing is needed, low intensity waves are used.

The structure of the solution in the zone of focusing of a weak shock wave is determined by the dissipation, although the equations which have to be solved do not contain viscous terms in the principal approximation. An analogous proposition was formulated in /5/, stating that the focusing zone can be described within the framework of linear non-viscous acoustics. Unlike the present work, in /5/ the effect of non-linearity instead of viscosity on the focusing of the sonic waves was investigated, and instead of the combining procedure the process of matching the solutions at a certain, completely specified distance from the focus, was used.

If the reflector or radiator has a boundary, then a diffraction wave will form in its neighbourhood and will arrive, in the linear formulation, at the focusing zone simultaneously with the focusing wave itself. In the present paper, the effect of the diffracted wave is neglected, as well as the effect of the penumbral zone. It is suggested that in order to determine the effect of this zone it is better to use the approach given in /6/ where it is suggested that the solution of geometrical non-viscous acoustics be combined with the solution of a certain diffusion equation describing the flow in the neighbourhood of the focus. The latter equation can be reduced, after some reduction, to the equation of non-linear sonic bundles /7/. However, no solution of the problem is given in /6/, and it is not at all clear how a solution can be obtained by confining oneself to two asymptotic zones without taking into account the penumbral zone away from the focusing zone.

The method of combining asymptotic expansions in a small parameter characterizing the weak compressibility of the medium which we use below, was used earlier to solve a number of one-dimensional problems /1, 8, 9/. Here it is generalized to the case of converging shock waves.

1. Formulation of the problem. Let us consider a reflector formed by part of an ellipsoid of revolution cut by a plane perpendicular to the axis of rotation. A spherically symmetrical explosion occurs at the focus which lies nearer to the reflector, and the explosion is modelled by a piston producing an initial velocity jump equal to $\bar{U}_0$, on a spherical surface of radius $\bar{r}_0$ with centre at the focus in question. Thus the problem is axisymmetric. Using the characeristic parameters: the characteristic temperature $\bar{T}_0$ and the density of the

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unperturbed medium $\bar\rho_0$, we introduce the dimensionless quantities

$$P = \frac{\bar P}{\bar\rho_0 \bar U_0^2}, \quad \mathbf{U} = \frac{\bar{\mathbf{U}}}{\bar U_0}, \quad x_k = \frac{\bar x_k}{\bar r_0}, \quad t = \frac{\bar t \bar U_0}{\bar r_0}, \quad T = \frac{\bar T}{\bar T_0},$$
$$s = \frac{\bar s \bar T_0}{\bar U_0^2}, \quad a = \frac{\bar a}{\bar r_0}, \quad b = \frac{\bar b}{\bar r_0}$$

Here $\bar P$ is the pressure, $\bar{\mathbf{U}}$ is the velocity vector in a Cartesian system of coordinate $\bar x_k$, $\bar t$ is time, $\bar T$ is the temperature, $\bar s$ is the entropy and $\bar a$ and $\bar b$ are the lengths of the semimajor and semiminor axes of the ellipsoid.

We will assume that the perturbations are such, that the density of the medium changes only slightly $\bar\rho = \bar\rho_0(1+\varepsilon\rho)$, where the small parameter $\varepsilon = (\partial P/\partial\bar\rho)_0\bar\rho_0$. Expanding the pressure $\bar P(\bar\rho, \bar s)$ in a series in $\bar\rho, \bar s$ about the unperturbed state and introducing the expansion coefficients k and k_1, we can write the system of equations of motion of a highly viscous fluid in the form

$$(1+\varepsilon\rho)\,d\mathbf{U}/dt = -\nabla\rho - k\varepsilon^2\rho\nabla\rho - k_1\nabla s + \eta\varepsilon^{-1/2}\Delta\mathbf{U} + \mu\varepsilon^{-1/2}\nabla\operatorname{div}\mathbf{U} \tag{1.1}$$
$$\varepsilon d\rho/dt + (1+\varepsilon\rho)\operatorname{div}\mathbf{U} = 0, \quad (1+\varepsilon\rho)\,T ds/dt = \varkappa\nabla T +$$
$$^1/_2\zeta\varepsilon^{-1/2}(\operatorname{div}\mathbf{U})^2 + {}^1/_2\mu\varepsilon^{-1/2}(\partial U_i/\partial x_k + \partial U_k/\partial x_i -$$
$$^2/_3\delta_{ik}\partial U_l/\partial x_l)^2; \quad d/dt = \partial/\partial t + \mathbf{U}\nabla$$
$$\zeta = \frac{\bar\zeta}{\bar\rho_0\bar U_0\bar r_0}\varepsilon^{1/2} \sim 1, \quad \mu = \frac{\bar\zeta + \bar\eta/3}{\bar\rho_0\bar U_0\bar r_0}\varepsilon^{1/2} \sim 1, \quad \varkappa = \frac{\bar\varkappa\bar T_0}{\bar\rho_0\bar U_0^3}$$

Here $\bar\zeta, \bar\eta$ are the coefficients of bulk and shear viscosity, and $\bar\varkappa$ is the thermal conductivity.

We will assume that the dimensions of the reflector are large compared with $\bar r_0$:

$$a = a^\circ\varepsilon^{-1/2} \sim \varepsilon^{-1/2}, \quad b = b^\circ\varepsilon^{-1/2} \sim \varepsilon^{-1/2}$$

The spherically symmetric shock wave which appears under the conditions given above is formed in a short time $t \sim \varepsilon^{1/2}$ (the time is measured from the instant at which a velocity jump appears), and, at times $t \sim 1$, has a Gaussian profile /1/

$$\rho = u_1\varepsilon^{-1/2} = G(t, r_1 + t\varepsilon^{-1/2}) \tag{1.2}$$
$$G(t,x) = \frac{\varepsilon^{-1/4}t^{-1/2}}{\sqrt{2\pi\alpha}}\exp\left[\frac{-x^2\varepsilon^{1/2}}{2t\alpha}\right], \quad \alpha = \zeta + \mu$$

where u_1, r_1 are the radial component of the velocity and the distance from the focus at which the explosion occurred.

Relation (1.2) shows that the width of the shock wave is of the order of $\varepsilon^{-1/4}$ by virtue of the high viscosity, while the characteristic spatial scale of the problem is $\varepsilon^{-1/2}$. Therefore, the wave (1.2) propagates as a locally hyperbolic wave, and until it reaches the reflector it will produce an exponentially small pressure and velocity perturbations in the neighbourhood of the reflector.

2. *Construction of the reflected wave.* We shall show that away from the reflector edges, at distances of a much higher order of magnitude than $\varepsilon^{-1/4}$, i.e. much greater than the wavelength, the wave (1.2) will be reflected like the usual wave in a non-viscous fluid. Let us introduce, near an arbitrary point M of reflector, away from its edges, the Cartesian coordinates x_M, y_M, z_M with origin at M. Then the zone of reflection will be characterized by the scales of the variables

$$x_M = x_M^\circ\varepsilon^{-1/4}, \quad y_M = y_M^\circ\varepsilon^{-1/4}, \quad z_M = z_M^\circ\varepsilon^{-1/4}, \quad \tau_M = (t - t_M)\,\varepsilon^{-1/4}$$
$$x_M^\circ, y_M^\circ, z_M^\circ, \tau_M \sim 1$$

where $t_M = r_{1M}\varepsilon^{1/2}$, r_{1M} is the distance between the first focus and the point M. The scales of the variables sought are $\rho \sim \varepsilon^{-1/4}$, $u \sim \varepsilon^{1/4}$.

We obtain, in the principal approximation, the usual system of equations of linear non-viscous acoustics for ρ, u. The asymptotic form of (1.2), in the zone in question, will take the form

$$\rho = u_1\varepsilon^{-1/2} = G(t_M, r_1 + t\varepsilon^{-1/2}) \tag{2.1}$$

We note that the asymptotic form (2.1) satisfies this system of equations.

The condition of adhesion imposed at the first boundary on the velocity of flow in a viscous fluid, will lead to the appearance of a boundary layer of width much smaller than $\varepsilon^{-1/4}$.

Outside this zone the wave will be reflected just as in the case of an ideal fluid, and, on moving away from the reflector, will take the form

$$\rho = -u_r\varepsilon^{-1/2} = G\,(t_M,\, r + t\varepsilon^{-1/2} - 2a) \tag{2.2}$$

where r is the distance to the second focus towards which the reflected wave is moving, u_r is the radial component of the velocity in a spherical system of coordinates with centre at the second focus, and the remaining velocity components are equal to zero.

Both (2.1) and (2.2) satisfy the equations of linear non-viscous geometrical acoustics, and their superposition also satisfies the zero boundary conditions for the normal component of the velocity. In the superposition of (2.1) and (2.2) the solution (2.1) becomes, as the time increases and on moving away from the reflector, exponentially small and only a single converging spherical wave (2.2) remains, with a certain distribution of the amplitude along its surface.

In what follows, we shall neglect the effect of the penumbral wave appearing as a result of reflection of the incident wave from a point at a distance of the order of $\varepsilon^{-1/4}$, from the edge of the reflector.

The solution (2.2) will hold when the wave moves away from the reflector by a distance Δr satisfying the condition $\varepsilon^{-1/4} \ll \Delta r \ll \varepsilon^{-1/2}$. It follows, therefore, that solution (2.2) represents an intermediate asymptotic form obtained when the solution describing the distribution of the parameters in the reflection zone $(\Delta r \sim \varepsilon^{-1/4})$, is combined with the solution of geometrical acoustics $(\Delta r \sim \varepsilon^{-1/2})$. The high viscosity of the medium means that it exerts its influence on the solution in the principal approximation.

3. *The construction of a converging reflected wave in the zone of geometrical acoustics.* Let us denote by $\mathbf{U}, x, y, z$ the velocity vectors and Cartesian coordinates attached to the second focus towards which the reflected wave travels. We take the scales of the variables in the form

$$x = x_0\varepsilon^{-1/2}, \quad y = y_0\varepsilon^{-1/2}, \quad z = z_0\varepsilon^{-1/2}, \quad t \sim 1$$
$$\mathbf{U} = \varepsilon^{1/4}\mathbf{U}_1(x_0, y_0, z_0, t, \theta) + \delta_2\mathbf{U}_2(x_0, y_0, z_0, t, \theta) + \ldots$$
$$\rho = \varepsilon^{-1/4}\rho_1 + \delta_2\varepsilon^{-1/4}\rho_2 + \ldots, \quad s \sim \max(\varepsilon, \varkappa)$$
$$\theta = \delta_s^{-1}\Phi(x_0, y_0, z_0, t)$$

Substituting these expansions into (1.1) we find that when the thermal conductivity is sufficiently small, the change in entropy is also small.

For a first approximation the system takes the form

$$\frac{\partial \mathbf{U}_1}{\partial t}\frac{\partial \Phi}{\partial t} = -\frac{\partial \rho}{\partial \theta}\nabla_0\Phi, \quad \frac{\partial \rho_1}{\partial \theta}\frac{\partial \Phi}{\partial t} + \frac{\partial \mathbf{U}_1}{\partial \theta}\nabla_0\Phi = 0 \tag{3.1}$$
$$\nabla_0 = \{\partial/\partial x_0, \partial/\partial y_0, \partial/\partial z_0\}$$

In the second approximation $(\delta_2 = \varepsilon^{1/2}, \delta_s = \varepsilon^{1/4})$

$$\frac{\partial \mathbf{U}_1}{\partial t} + \frac{\partial \mathbf{U}_2}{\partial \theta}\frac{\partial \Phi}{\partial t} = -\nabla_0\rho_1 - \frac{\partial \rho_2}{\partial \theta}\nabla_0\Phi + \eta\frac{\partial^2 U_1}{\partial \theta^2}\nabla_0^2\Phi + \tag{3.2}$$
$$\mu\left(\frac{\partial^2 \mathbf{U}_1}{\partial \theta^2}\nabla_0\Phi\right)\nabla_0\Phi$$
$$\frac{\partial \rho_1}{\partial t} + \frac{\partial \rho_2}{\partial \theta}\frac{\partial \Phi}{\partial t} + \operatorname{div}_0\mathbf{U}_1 + \frac{\partial \mathbf{U}_2}{\partial \theta}\nabla_0\Phi = 0$$

From (3.1) it follows that

$$|\nabla_0\Phi|^2 - (\partial\Phi/\partial t)^2 = 0, \quad \mathbf{U}_1 = \lambda(x_0, y_0, z_0, t, \theta)\nabla_0\Phi, \quad \rho_1 = -\lambda\partial\Phi/\partial t$$

Determining λ from the condition of compatibility of the system (3.2), we obtain

$$2\frac{\partial \lambda}{\partial t}\frac{\partial \Phi}{\partial t} - 2\nabla_0\lambda\,\nabla_0\Phi - \Delta_0\Phi = \alpha\left(\frac{\partial \Phi}{\partial t}\right)^3\frac{\partial^2\lambda}{\partial \theta^2} \tag{3.3}$$

The reflected wave has a spherical form (2.2) near the reflector, and it is convenient to introduce here the spherical coordinates

$$x_0 = r_0\cos\varphi\cos\psi, \quad y_0 = r_0\sin\varphi\cos\psi, \quad z_0 = r_0\sin\psi$$

Then, taking into accout the initial condition, we shall have $\Phi = -r_0 - t$. From (3.3) we obtain, in terms of the variable $\Lambda = \lambda r_0$, $\xi_0 = t + r_0$, $\eta_0 = t - r_0$,

$$\partial\Lambda/\partial\eta_0 - {}^1/_4\alpha\partial^2\Lambda/\partial\theta^2 = 0 \tag{3.4}$$

Let us combine the solution within the wave reflection some with the solution of (3.4) which we shall take in the form

$$\Lambda = \Lambda_0(\psi)[\eta_0 + \eta_0^\circ(\psi)]^{-1/2} \exp\left[\frac{-(\theta + \theta_0(\psi))^2}{\alpha(\eta + \eta_0^\circ(\psi))}\right] \tag{3.5}$$

In the intermediate zone the asymptotic form of the solution has the form (2.2). After combining the solutions we find that

$$\Lambda_0(\psi) = t_M^{-3/2} r_{0M} \sqrt{(t_M - r_{0M} + 2a^\circ)/(2\pi\alpha)} \tag{3.6}$$
$$\theta_0 = -2a^\circ, \quad \eta_0^\circ = 2a^\circ, \quad t_M = t_M(\psi), \quad r_{0M} = r_{0M}(\psi)$$

Taking into account the fact that outside the region $\xi_0 - 2a^\circ \sim \varepsilon^{1/4}$ the right-hand side of the first equation of (3.6) is exponentially small, we finally obtain the asymptotic form in the zone of geometrical acoustics

$$\rho = \frac{\varepsilon^{-1/4}\Lambda_0(\psi)}{r_0\sqrt{2(a^\circ - r_0)}} \exp\left[\frac{-(\xi_0 - 2a^\circ)^2}{2\alpha\varepsilon^{1/2}(2a^\circ - r_0)}\right] \tag{3.7}$$

In the neighbourhood of the second focus $r_0 \to 0$ and solution (3.7) has a singularity. This implies the need to derive another asymptotic expansion at this point.

4. Construction of a solution in the focusing zone. The scales of the variables in this zone are:

$$x = x_*\varepsilon^{-1/4}, \quad y = y_*\varepsilon^{-1/4}, \quad z = z_*\varepsilon^{-1/4}, \quad t = 2a^\circ + t_*\varepsilon^{1/4},$$
$$\mathbf{U} = \varepsilon_*\mathbf{U}_*, \quad \rho = \varepsilon_*\varepsilon^{-1/2}\rho_*$$

In the principal approximation we have

$$\partial\mathbf{U}_*/\partial t_* = -\nabla_*\rho_*, \quad \partial\rho_*/\partial t_* + \operatorname{div}_*\mathbf{U}_* = 0 \tag{4.1}$$

Changing to the variables $\xi_* = t_* + r_*$, $r_* = \sqrt{x_*^2 + y_*^2 + z_*^2}$, ψ, φ, we obtain, for the axisymmetric flow,

$$\frac{\partial^2\rho_*}{\partial r_*^2} + \frac{2}{r_*}\frac{\partial\rho_*}{\partial r_*} + \frac{1}{r_*^2\sin\psi}\frac{\partial}{\partial\psi}\left(\sin\psi\frac{\partial\rho_*}{\partial\psi}\right) = -\frac{2}{r_*}\frac{\partial}{\partial r_*}\left(r_*\frac{\partial\rho_*}{\partial\xi_*}\right)$$

Separating the variables $\rho_* = Y(r_*, \psi)Y_1(\xi_*)$, $Y = R(r_*)\Omega(\psi)$ and taking one of the eigenvalues as complex, we obtain

$$\frac{dY_1}{d\xi_1} + iaY_1 = 0, \quad \frac{1}{\sin\psi}\frac{d}{d\psi}\left(\sin\psi\frac{d\Omega}{d\psi}\right) + q\Omega = 0 \tag{4.2}$$
$$\frac{d}{dr_*}\left(r_*^2\frac{dR}{dr_*}\right) - 2air_*\frac{d}{dr_*}(r_*R) - qR = 0$$

The solution will be bounded only when $q = n(n+1)$, where n is an integer. In the general solution (4.2) we must also have only functions that are bounded as $r_* \to 0$. As a result, we can seek the solution near the focus in the form

$$\rho_* = \frac{1}{\sqrt{2r_*}}\operatorname{Real}\left\{\sum P_n(\cos\psi)\int C_n(\omega)\exp(-i\omega t_*)J_{\nu_n}(r_*\omega)\,d\omega\right\} \tag{4.3}$$
$$\nu_n = \sqrt{q_n + 1/4}, \quad q_n = n(n+1)$$

where P_n are spherical functions, J_{ν_n} is a Bessel function, the summation is carried out from $n = 0$ to $n = \infty$, and the integration in ω is from $-\infty$ to $+\infty$.

The above expressions can also be obtained more rapidly from the initial system.

Let us combine (4.3) and (3.7). The intermediate asymptotic form (3.7) is obtained in the form $(r_0 \to 0)$

$$\rho = \frac{\varepsilon^{-3/4}\Lambda_0(\psi)}{2\sqrt{a^\circ}\,r}\exp\left(-\frac{\xi_*^2}{4\alpha a^\circ}\right) \tag{4.4}$$

and the intermediate asymptotic form (4.3) $(r_* \to \infty)$ is

$$\rho = \frac{\varepsilon_* \varepsilon^{-1/2}}{2\sqrt{\pi}} \frac{1}{r_*} \operatorname{Real} \Big\{ \sum P_n(\cos\psi) \Big[\int \frac{C_n(\omega)}{\sqrt{\omega}} \exp(-i\omega\xi_*) \times \exp\Big(i\Big(\frac{\pi\nu_n}{2} + \frac{\pi}{4}\Big)\Big) d\omega + \int \frac{C_n(\omega)}{\sqrt{\omega}} \exp\Big(-i\Big(\frac{\pi\nu_n}{2} + \frac{\pi}{4}\Big)\Big) \times \exp(-i\omega(t_* - r_*))\, d\omega \Big] \Big\} \tag{4.5}$$

In the second integral of (4.5) we take the asymptotic form with $\xi_* = \text{const}$, $t_* \to -\infty$, $r_* \to \infty$, i.e. the exponent of the second integral tends to $-i\omega\infty$. Therefore, the principal term of the asymptotic form contains only the first integral. From the matching condition we see that we must take $C_n(\omega) = \sqrt{\omega} f(\omega) B_n$, where B_n is independent of ω, $\varepsilon_* = 1$. Matching will occur if

$$\sum P_n(\cos\psi) \exp(i(\pi\nu_n/2 + \pi/4)) B_n = \sqrt{\pi/a^\circ}\, \Lambda_0(\psi)$$

From this it follows that

$$B_n = -\sqrt{\frac{\pi}{a^\circ}} \frac{2n+1}{2} \exp\Big(-i\Big(\frac{\pi\nu_n}{2} + \frac{\pi}{4}\Big)\Big) \int_0^\pi \Lambda_0(\psi) P_n(\cos\psi) \sin\psi\, d\psi$$

$$f(\omega) = \sqrt{\alpha a^\circ/\pi} \exp(-\alpha a^\circ \omega^2)$$

Finally, near the focus the solution will take the form

$$\rho = \sqrt{\frac{\alpha a^\circ}{2\varepsilon\pi}} \operatorname{Real} \Big\{ \sum P_n(\cos\psi) B_n \int \sqrt{\omega} \exp(-\alpha a^\circ \omega^2 - i\omega t_*) \frac{J_{\nu_n}(r_*\omega)}{\sqrt{r_*}}\, d\omega \Big\} \tag{4.6}$$

5. ***Determination of the pressure at the focus.*** Taking into account the fact that $J_{\nu_n}(r_*\omega)/\sqrt{r_*} \to \sqrt{\omega}\Gamma(\nu_0 + 1)$ when $n = 0$ and that the limit is equal to zero when $n \neq 0$, we can obtain from (4.6) at the focus

$$\rho_f(t_*) = -K(t_*) \int_0^\pi \Lambda_0(\psi) \sin\psi\, d\psi$$
$$K(t_*) = (4\varepsilon^{1/2} a^{\circ 3/2} \alpha)^{-1} t_* \exp(-t_*^2/(4a^\circ\alpha)) \tag{5.1}$$

If we assume that an intermediate asymptotic zone exists in which the wave profile is (4.4) and is described at the same time by system (4.1), we can specify the initial conditions when $t_* = t_*^\circ < 0$ $(r_* = -t_*^\circ)$:

$$\rho|_{t_*^\circ} = B \exp\Big[-\frac{(t_*^\circ + r_*)^2}{4a^\circ\alpha}\Big], \quad B = -\frac{\varepsilon^{1/2}\Lambda_0(\psi)}{2\sqrt{a^\circ}\, t_*^\circ}$$
$$\frac{\partial\rho}{\partial t_*}\Big|_{t_*^\circ} = -\frac{B}{2a^\circ\alpha}(t_*^\circ + r_*) \exp\Big[-\frac{(t_*^\circ + r_*)^2}{4a^\circ\alpha}\Big] \tag{5.2}$$

Then, using Poisson's formula, we obtain an expression differing from (5.1) in having a multiplying factor $[1 + (a^\circ\alpha/t_* - t_*)/t_*^\circ]$ on the right-hand side. However, taking into account the fact that the initial conditions can be taken in the form (5.2) only outside the focusing zone, we can pass in the latter expression to the limit as $t_*^\circ \to -\infty$. This will yield relation (5.1). Thus we see that both methods of determining the dependence of the pressure on time at the focus are asymptotically equivalent. It is difficult to use Poisson's formula near the focus; we must therefore use the solution in the form (4.6).

The function $\Lambda_0(\psi)$ depends on the reflecting system. For the case in question we have

$$\Lambda_0(\psi) = [\sqrt{\pi\alpha}(1 - 2\delta\cos\psi + \delta^2)]^{-1}, \quad \delta = \sqrt{a^2 - b^2}/a$$

If the reflector represents part of an ellipsoid with an aperture angle of ψ_0, we can obtain the following expression for the pressure at the focus:

$$\rho_f(t_*) = \frac{K(t_*)}{2\sqrt{\pi}}\, g(\delta), \quad g(\delta) = \frac{1}{\delta}\ln\frac{1+\delta^2-2\delta\cos\psi_0}{(1-\delta)^2} \tag{5.3}$$

From (5.3) we obtain, as $\delta\to 0$, the dependence of the pressure on time, at the focus, for a spherical reflector

$$\rho_f(t_*) = -\frac{K(t_*)}{2\sqrt{\pi}}(1-\cos\psi_0) \tag{5.4}$$

Taking $\psi_0 = \pi$, we obtain a relationship at the focus for the reflection from a complete ellipsoid or complete sphere. In the latter case we obtain the solution of the problem of the focusing of a spherically symmetrical wave.

Thus we see from (5.1) that the wave has an antisymmetric profile at the focus. The compression wave is followed by a rarefaction wave of the same form. We see that the maximum and minimum pressures at the focus are reached at $t_* = \mp\sqrt{2a^\circ\alpha}$ and are equal in modulus to

$$\rho_{f\max} = g(\delta)\,\varepsilon^{-1/4}/(4\sqrt{2\pi e}\,a^\circ\alpha)$$

The dependence of the wave amplitude on the eccentricity and aperture angle of the reflector is given by the function shown in the figure for various aperture angles of the ellipsoid (curves *1*-*5* correspond to the angles $\psi_0 = \pi,\ 5\pi/6,\ \pi/2,\ \pi/3,\ \pi/6$). When $\delta\to 1$, all the curves tend logarithmically to infinity. When the angle ψ_0 decreases, so does the amplitude of the wave. This decrease may be compensated by increasing the eccentricity of the ellipsoid. Although the dependence of the amplitude on the eccentricity has a monotonic form for each aperture angle, the increase $\partial g/\partial\delta$ itself depends on ψ_0 non-monotonically. When the values of ψ_0, are close to π and zero, there is a slight increase in amplitude for small δ, but within the interval between π and zero the increase becomes larger (see e.g. curve *3* whose slope for small δ is greater than that of curves *1* and *5*).

6. Conclusions. The use of the method of combined asymptotic expansions has enabled us to obtain an analytic solution of the problem over the whole space-time domain. The possibility of constructing composite asymptotic forms is, however, restricted by the behaviour of the expansions in adjacent zones. The solution of geometrical acoustics in the focusing zone tends to infinity. This implies that the additive composite expansion acquires, at the focus, a redundant term equal to

$$\varepsilon^{-1/4}\Lambda_0(\psi)\frac{\partial}{\partial r_0}\left[\sqrt{2(2a^\circ-r_0)}\exp\left(\frac{-\xi^2}{2\alpha\varepsilon^{1/2}(2a^\circ-r_0)}\right)\right]\Bigg|_{r_0=0}$$

From this it follows that we cannot construct a composite expansion which could be used in the geometrical acoustics zone and in the focusing zone.

The presence of viscosity means that the shock wave becomes, by the time it arrives at the focus, so diffuse, that the focusing diffraction zone can be described in the principal approximation by the equation of an ideal fluid.

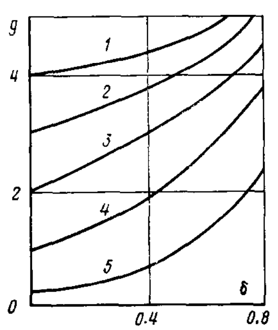

Fig.1

The shock wave caused by expansion, which in a highly viscous fluid has a Gaussian profile, arrives at the focus in the form of an antisymmetric perturbation, and its amplitude may have increased either by virtue of the increase in the aperture angle ψ_0 of the ellipsoid (keeping the lengths of the semi-axes fixed), or by virtue of increased eccentricity.

However, as ψ_0 decreases, the penumbral zone and the wave diffracted from the edges will both assume increasing importance. The gometrical acoustics approximation will cease to hold near the edge rays $(\psi = \psi_0)$.

The pressure at the focus itself as well as in its neighbourhood will depend on the viscosity only in the combination $\alpha a^\circ = \bar U_0\bar a(2\bar\zeta + \bar\eta/3)/(\bar C_0^2\bar r_0^2\bar\rho_0)$, where $\bar C_0$ is the velocity of sound in an unperturbed medium. The maximum and minimum pressures at the focus are inversely proportional to this quantity.

The solution (4.6) obtained can be used to calculate other focusing shock waves with spherical phase surfaces, and in particular for the focusing wave obtained after a plane shock wave is reflected from a parabolic reflector.

Although the problem of the effect of diffraction waves has not been discussed, it is clear that the solution obtained here will be worse, the smaller the aperture angle of the focusing wave. On the other hand, when the reflector has the form of a complete ellipsoid and the focusing wave has the form of a complete sphere, the solution will have not have this shortcoming, since in this case we have neither a diffracted wave, nor a penumbral zone. In this case we cannot obtain any diffusion equations within the focusing zone. These two limiting cases, namely the case of a small aperture angle of the reflector discussed in /6/, and a fully opened reflector, lead to basically different structures of the solution within the focusing zone. We can obtain a solution suitable for all cases only by taking into account the penumbral zone away from the focus. We note that the methods used in /10/ do not enable the effect of the reflector edges on the focusing of the shock wave in a viscous fluid to be taken into account, since they are based on the wave acoustics of ideal media.

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APPROXIMATE FORMULAS FOR HEAT FLOWS TOWARDS AN IDEALLY CATALYTIC SURFACE NEAR A PLANE OF SYMMETRY*

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The three-dimensional flow of a chemically unstable viscous gas near a plane of symmetry of blunt bodies streamlined at the angle of attack, is considered. The investigation is carried out using a model of a thin, viscous shock layer. To a first approximation of the method of successive approximations for a uniform gas simple formulas are obtained for the distribution of the heat flux over the surface, referred to its value at the stagnation point. It is shown that for a chemically unstable gas the distribution of the heat flux along an ideally catalytic surface depends only slightly on the conditions prevailing within the incident flow, is determined mainly by the geometrical characteristics of the body, and is described quite satisfactorily by the formulas obtained. The accuracy of these formulas is determined by comparison with numerical computations carried out for bodies of various shapes, moving at different angles of attack along a planing trajectory of re-entry into the Earth's atmosphere, and during re-entry into the atmosphere at a constant velocity.

1. Let us consider the three-dimensional steady flow past a blunt body of a stream of

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